

THICK TRIANGULATIONS OF HYPERBOLIC n -MANIFOLDS

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ABSTRACT. We show that a complete hyperbolic n -manifold has a geodesic triangulation such that the tetrahedra contained in the thick part are L -bilipschitz diffeomorphic to the standard Euclidean n -simplex, for some constant L depending only on the dimension and the constant used to define the thick-thin decomposition of M .

A geodesic triangulation of a complete hyperbolic n -manifold M may be forced by the geometry of M to have simplices which are either small or flat. Big simplices without small dihedral angles cannot live in the thin part of M . We show that a complete hyperbolic n -manifold has a geodesic triangulation such that the tetrahedra contained in the thick part are L -bilipschitz diffeomorphic to the standard Euclidean n -simplex, for some constant L depending only on the dimension n and the constant μ used to define the thick-thin decomposition of M . We call such a triangulation a (μ, L) -thick geodesic triangulation of M .

Theorem 1. *Let $n \geq 2$. Let μ be a Margulis constant for \mathbb{H}^n . There exists a constant $L := L(n, \mu)$ such that every complete hyperbolic n -manifold has a (μ, L) -thick geodesic triangulation.*

To prove existence of thick geodesic triangulations, we examine Delaunay triangulations of well-spaced point sets in hyperbolic n -space and the problem of eliminating flat simplices (i.e. simplices with small dihedral angles). The corresponding question for 3-dimensional Euclidean space has been well-studied. The only tetrahedra in such a triangulation which can have small dihedral angles are called *slivers*, and it was a problem to show how to remove them without creating new ones. Several techniques for removing slivers have been developed in the Euclidean setting (see [ELM⁺00], [MTTW96], [Li00], [Li03]). We adapt the technique introduced in [ELM⁺00] of perturbing vertices of a Delaunay triangulation in order to remove slivers to the hyperbolic setting.

Existence of thick geodesic triangulations implies that any hyperbolic n -manifold has a geodesic triangulation such that the tetrahedra contained in the thick part come from some fixed compact set of tetrahedra (which does not depend on the manifold). We use this compactness to prove existence of universal bounds on the principal curvatures of certain surfaces in hyperbolic 3-manifolds in [Bre06]. See [Kap07] for another application of thick triangulations.

Emil Saucan has shown that hyperbolic n -orbifolds have triangulations whose simplices are uniformly round (called “fat” triangulations), and he uses this to prove existence of quasi-meromorphic maps which are automorphic with respect to the corresponding Kleinian group (See [Sau06a], [Sau06b], [Sau05]). However, the triangulations he produces have no uniform bound on the size (i.e. edge lengths) of the simplices, even in the thick part.

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Definition (*thick-thin decomposition*). Let $\mu > 0$. The μ -thick part of M , denoted by $M_{[\mu, \infty)}$ is the set of points where the injectivity radius is at least $\mu/2$. The μ -thin part of M , denoted by $M_{(0, \mu]}$, is the closure of the complement of $M_{[\mu, \infty)}$.

Definition (*thick triangulation*). Let $\mu > 0$, $L > 0$. A triangulation T of a complete hyperbolic n -manifold M is (μ, L) -thick if every n -simplex of T which is contained in the μ -thick part of M is L -bilipschitz diffeomorphic to the standard Euclidean n -simplex. Once we have fixed μ and L , we will just refer a *thick* triangulation.

We want to find triangulations such that the simplices in the thick part of M are neither too big nor too small, and which do not have small dihedral angles. It is not difficult to find triangulations such that the simplices in the thick part are neither too big nor too small. Let $\mu > 0$ be a Margulis constant for \mathbb{H}^n . Let $\epsilon := \mu/100$, $\delta := \epsilon/10$. Let \mathcal{S} be a generic set of points in a complete hyperbolic n -manifold M such that for any $p \in M$ the ball $B(p, \text{inj}(M, p)/5)$ centered at p with radius $\text{inj}(M, p)/5$ contains a point of \mathcal{S} in its interior. Also assume that the set \mathcal{S} is maximal with respect to the condition that each point in $\mathcal{S} \cap M_{[\mu, \infty)}$ is no closer than ϵ to another point of \mathcal{S} . Let \tilde{T} be the Delaunay triangulation of \mathcal{S} . Any simplex of \tilde{T} in the μ -thick part of M has edge lengths in the interval $[\epsilon, 2\epsilon]$. In fact, this triangulation is

not very far from the one we want. We will show that each vertex of $S \cap M_{[\mu, \infty)}$ can be moved a small distance so that the Delaunay triangulation of the new set of points is (μ, L) -thick.

Definition (*altitude*). The *altitude* of a vertex v of a geodesic n -simplex in \mathbb{H}^n is the distance from v to the hyperplane of \mathbb{H}^n containing the other vertices.

Definition (*good simplices*). For $2 \leq k \leq n$, $0 < a < b$, $0 < d$, a geodesic k -dimensional simplex S in hyperbolic n -space \mathbb{H}^n is (a, b, d) -good if the lengths of the edges of S are contained in the interval $[a, b]$ and the altitude of each vertex of S is at least d . When a, b, d are understood, we will refer to *good simplices*. We say S is *bad* if it is not good.

Remark 1. If a geodesic n -simplex S in \mathbb{H}^n has edge lengths in $[a, b]$, then there are two ways that it can be (a, b, d) -bad for a small number $d > 0$. Either S has big circumradius or the vertices of S all lie near a hyperbolic $(n-2)$ -sphere. In the triangulation \tilde{T} described above the simplices in $\tilde{T} \cap M_{[\mu, \infty)}$ have bounded circumradii, so that the vertices of any (a, b, d) -bad simplex in $\tilde{T} \cap M_{[\mu, \infty)}$ must all lie very close to a hyperbolic $(n-2)$ -sphere (The vertices get closer to a sphere as $d \rightarrow 0$).

Remark 2. Let $b > a > 0$ and $d > 0$. Consider the set of compact hyperbolic n -simplices in \mathbb{H}^n up to isometry. The set of geodesic (a, b, d) -good simplices is a compact subset. Thus we have the following lemma.

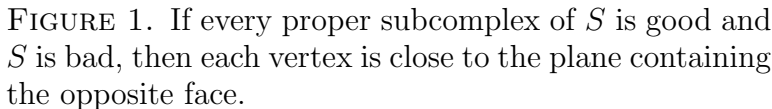
Lemma 1. *For each $n \geq 2$, $b > a > 0$, $d > 0$, there exists a constant $L := L(n, a, b, d)$ such that each (a, b, d) -good simplex is L -bilipschitz diffeomorphic to the standard Euclidean n -simplex.*

Definitions (*Delaunay triangulation*). Let \mathcal{S} be a generic set of points in M such that for any $p \in M$ the ball $B(p, \text{inj}(M, p)/5)$ centered at p with radius $\text{inj}(M, p)/5$ contains a point of \mathcal{S} in its interior. The *Delaunay triangulation* of \mathcal{S} is the geodesic triangulation of M determined as follows: A set, $\{p_1, \dots, p_{n+1}\}$, of $n+1$ vertices in \mathcal{S} determines an n -simplex in \mathcal{T} if and only if the minimal radius circumscribing sphere contains no points of \mathcal{S} in its interior.

Definition (*good perturbation*). Let $\delta := \delta(\mu) \leq \epsilon(\mu)/10$. A δ -good perturbation of \mathcal{S} is a collection of points \mathcal{S}' in M such that there exists a bijection $\phi : \mathcal{S} \rightarrow \mathcal{S}'$ with $d(p, \phi(p)) \leq \delta$ for every $p \in \mathcal{S}$. Denote $\phi(p)$ by p' . If \mathcal{T} and \mathcal{T}' are the Delaunay triangulations of \mathcal{S} and \mathcal{S}' , then we

Definition (*bad region*). Let $S = [v_1, \dots, v_k]$ be a geodesic $(k - 1)$ -simplex in \mathbb{H}^n . Let $b > a > 0$, $c > 0$, $d > 0$. The (a, b, c, d) -*bad region of T* is the set of points p in \mathbb{H}^3 such that $[p, v_1, \dots, v_k]$ has edge lengths in $[a, b]$ and circumradius at most c , and the distance from p to the hyperplane containing the opposite face is less than d .

Lemma 2. *Let $S = [v_0, \dots, v_k]$ be a geodesic k -simplex in \mathbb{H}^k with edge lengths in $[a, b]$. If every proper subcomplex of S is (a, b, d) -good and S is (a, b, d) -bad, then the distance from each vertex of S to the hyperplane containing the opposite face is at most a constant $D := D(b, d_0, d)$ such that $D(b, d_0, d) \rightarrow 0$ as $d \rightarrow 0$ and b and d_0 remain fixed.*



Proof. Since S is (a, b, d) -bad and the edge lengths are in $[a, b]$, the distance from some vertex, say v_0 , to the hyperplane in \mathbb{H}^k containing the opposite face $[v_1, \dots, v_k]$ is less than d . Let P_0 be the hyperplane containing v_0, \dots, v_{k-1} . Let P_k be the hyperplane containing v_1, \dots, v_k . Let α be the angle between P_0 and P_k . Let v'_0 be the orthogonal

projection of v_0 to $P_0 \cap P_k$ and let v_0'' be the orthogonal projection of v_0 to P_k . The angle of the hyperbolic triangle $[v_0, v_0', v_0'']$ at v_0' is α . See figure 4(a). The hyperbolic law of sines [Fen89] gives us

$$\sin(\alpha) = \frac{\sinh(\|[v_0, v_0'']\|)}{\sinh(\|[v_0, v_0']\|)}.$$

Let v_k' be the orthogonal projection of v_k to $P_0 \cap P_k$ and let v_k'' be the orthogonal projection of v_k to P_0 . The angle of the hyperbolic triangle $[v_k, v_k', v_k'']$ at v_k' is also α . See figure 4b. Using the hyperbolic law of sines again we get

$$\begin{aligned} \sinh(\|[v_k, v_k'']\|) &= \sin(\alpha) \cdot \sinh(\|[v_k, v_k']\|) \\ &= \frac{\sinh(\|[v_0, v_0'']\|)}{\sinh(\|[v_0, v_0']\|)} \cdot \sinh(\|[v_k, v_k']\|). \end{aligned}$$

Now $\|[v_0, v_0']\| \geq d_0$ and $\|[v_k, v_k']\| \leq b$ since $\|[v_0, \dots, v_{k-1}]\|$ and $\|[v_1, \dots, v_k]\|$ are (a, b, d_0) -good $(k-1)$ -simplices. Also, $\|[v_0, v_0'']\| < d$ by our assumption. Thus we have

$$\sinh(\|[v_k, v_k'']\|) \leq \frac{\sinh(d)}{\sinh(d_0)} \cdot \sinh(b).$$

We have shown that the distance from v_k to the hyperplane containing $[v_0, \dots, v_{k-1}]$ is at most

$$D(b, d_0, d) := \operatorname{asinh}\left[\frac{\sinh(d)}{\sinh(d_0)} \cdot \sinh(b)\right].$$

A similar argument shows the distance from each vertex to the hyperplane containing the opposite face is at most

$$D(b, d_0, d).$$

□

Next we show that if a bad k -simplex has bounded circumradius and good proper simplices, then the vertices lie near hyperbolic $(k-2)$ -spheres.

Lemma 3. *Let $S = [v_0, \dots, v_k]$ be a geodesic k -simplex in \mathbb{H}^k with edge lengths in $[a, b]$ and circumradius at most c . If every proper subcomplex of S is (a, b, d_0) -good and S is (a, b, d) -bad, then the distance from each vertex to the circumsphere of the opposite face is at most a constant $R := R(a, b, c, d_0, d)$. Moreover, R can be chosen so that $R \rightarrow 0$ as $d \rightarrow 0$ and k, a, b, c remain fixed.*

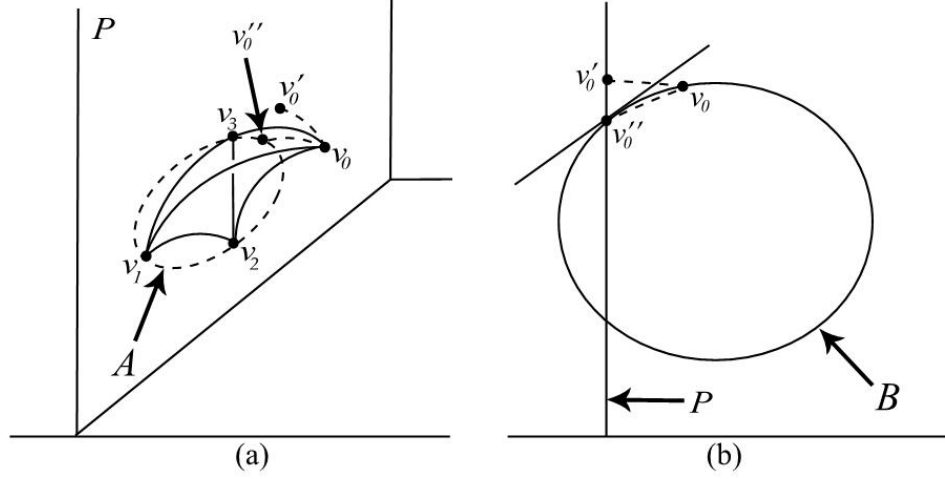


FIGURE 2. The distance from v_0 to the circumsphere A of $[v_1, \dots, v_k]$ is small if every proper subcomplex of S is good, but S is bad

Proof. Since every proper subcomplex of S is (a, b, d_0) -good and S is (a, b, d) -bad and has edge lengths in $[a, b]$ and circumradius at most c , the distance from each vertex to the hyperplane containing the opposite face is at most the constant $D := D(b, d_0, d)$ provided by Lemma 2.

Let A be the circumsphere of $[v_1, \dots, v_k]$. Let B be the circumsphere of S . Now A is the intersection of B with some hyperplane P . Let v'_0 be the orthogonal projection of v_0 to P . Let v''_0 be the point on A which is closest to v_0 . Let Q be the 2-dimensional hyperbolic plane which contains v_0 , v'_0 , and v''_0 . Now $B \cap Q$ is a hyperbolic circle which intersects the hyperbolic line $P \cap Q$ (see figure 2). Since the radii of A and B are in the interval $[a/2, c]$, the angle between $P \cap Q$ and the tangent of $B \cap Q$ at the two points in $(P \cap Q) \cap (B \cap Q)$ is bounded from below by a positive constant $\alpha_0 := \alpha_0(a, c)$. Thus the angle α between $[v''_0, v_0]$ and $[v'_0, v_0]$ is bounded from below by α_0 . Let β be the angle between $[v_0, v'_0]$ and $[v_0, v''_0]$. By the hyperbolic law of sines we have

$$\begin{aligned} \sinh(\|[v'_0, v''_0]\|) &= \frac{\sinh(\|[v_0, v'_0]\|)}{\sin(\alpha)} \cdot \sin(\beta) \\ &\leq \frac{\sinh(D)}{\sin(\alpha_0)}. \end{aligned}$$

The triangle inequality now gives us

$$\begin{aligned} ||[v_0, v_0'']|| &\leq ||[v_0, v_0']|| + ||[v_0', v_0'']|| \\ &\leq D(b, d_0, d) + \operatorname{asinh}\left(\frac{\sinh(D(b, d_0, d))}{\sin(\alpha_0(a, c))}\right). \end{aligned}$$

Let $R(a, b, c, d_0, d) := D(b, d_0, d) + \operatorname{asinh}\left(\frac{\sinh(D(b, d_0, d))}{\sin(\alpha_0(a, c))}\right)$.

□

Lemma 4. *Let $b > a > 0, c > 0, d_0 > 0, d > 0$. Let $n \geq 3$ and $k < n$. Let S be a geodesic k -simplex in \mathbb{H}^n such that the circumradius of S is at most c and every proper sub-simplex of S is (a, b, d_0) -good. The volume of the (a, b, c, d) -bad region of S is at most a constant $V_k := V_k(n, a, b, c, d_0, d)$ such that $V_k \rightarrow 0$ as $d \rightarrow 0$ and a, b, c, d_0 remain fixed.*

Proof. The (a, b, c, d) -bad region of S is contained in the R -neighborhood of the circumsphere B of S , where $R := R(a, b, c, d_0, d)$ is the constant provided by Lemma 3. Since the radius of S is at most c and $R(a, b, c, d_0, d) \rightarrow 0$ as $d \rightarrow 0$, we can let V_k be the volume of the R -neighborhood of a $(k-1)$ -dimensional hyperbolic sphere of radius c in \mathbb{H}^n . □

Proof of Theorem. The idea of the proof of Theorem is to perturb the vertices of a Delaunay triangulation of M so that every n -simplex contained in the thick part of M is in $G(n, a, b, c, d)$ for some fixed a, b, c, d .

Let $\mu > 0$ be a Margulis constant for \mathbb{H}^n . Let $\epsilon := \mu/100$, $\delta := \epsilon/10$. Let \mathcal{S} be a generic set of points in a complete hyperbolic n -manifold M such that for any $p \in M$ the ball $B(p, \operatorname{inj}(M, p)/5)$ centered at p with radius $\operatorname{inj}(M, p)/5$ contains a point of \mathcal{S} in its interior. Also assume that the set \mathcal{S} is maximal with respect to the condition that each point in $\mathcal{S} \cap M_{[\mu, \infty)}$ is no closer than ϵ to another point of \mathcal{S} . Let T be the Delaunay triangulation of \mathcal{S} . We assume that ϵ is sufficiently small with respect to the injectivity radius so that if we perturb a point $p \in \mathcal{S}$, any changes in T will occur in a ball which lifts to the universal cover. Thus we may work in \mathbb{H}^n . This maximality condition on \mathcal{S} also implies that each k -dimensional simplex S in the Delaunay triangulation of \mathcal{S} which is contained in the ϵ -thick part of M has edge lengths in the interval $[\epsilon, 2\epsilon]$ and circumradius at most ϵ (for $k = 1, \dots, n$). The only way a k -simplex in T can be unfair is to have a vertex which is very close to the hyperplane containing the opposite face. Note that if t is a k -simplex in the Delaunay triangulation of a δ -good perturbation

of \mathcal{S} which is contained in $M_{[\mu, \infty)}$, then t has edge lengths between $\epsilon - 2\delta$ and $2\epsilon + 2\delta$, and circumradius no more than $\epsilon + \delta$. Thus the only bad k -simplices are still those with a vertex which is very close to the hyperplane containing the opposite face.

We need two more lemmas before finishing the proof of the theorem.

Lemma 5. *Let $p \in \mathcal{S} \cap M_{[\mu, \infty)}$. For each $k = 3, \dots, n$, the number of k -tuples $\{v_1, \dots, v_k\}$ such that $[p', v'_1, \dots, v'_k]$ is a k -simplex in some δ -good perturbation of T is bounded by a constant $N := N(n, k, \mu)$.*

Proof. The $(\frac{\epsilon}{2} - \delta)$ -balls centered at the points of $\mathcal{S} \cap M_{[\mu, \infty)}$ are mutually disjoint since no two points of $\mathcal{S} \cap M_{[\mu, \infty)}$ are closer than $\epsilon - 2\delta$ to each other. If p' and q' are vertices of a k -simplex in a δ -good perturbation \mathcal{T}' of \mathcal{T} , then $d(p', q') \leq 2\epsilon + 2\delta$, so that the $(\frac{\epsilon}{2} - \delta)$ -ball centered at q' is contained in the $(2\epsilon + 2\delta)$ -ball centered at p' . There can be at most

$$m := m(n, \mu) = \left\lceil \frac{\text{vol}_{\mathbb{H}^n}(B(2\epsilon + 2\delta))}{\text{vol}_{\mathbb{H}^n}(B(\frac{\epsilon}{2} - \delta))} \right\rceil$$

mutually disjoint $(\frac{\epsilon}{2} - \delta)$ -balls contained in a $(2\epsilon + 2\delta)$ -ball, where $[w]$ is the integer part of w . One of these is centered at p' . So there are at most $m - 1$ vertices in \mathcal{S} which may be the vertex of a k -simplex in \mathcal{T}' which also has p' as a vertex. Thus the number of k -tuples $\{v_1, \dots, v_k\}$ of points in \mathcal{S} such that $[p', v'_1, \dots, v'_k]$ is a k -simplex in some good perturbation of \mathcal{T} is at most $\binom{m}{k}$. Let $N(n, k, \mu) := \binom{m(n, \mu)}{k}$. \square

We will remove the bad simplices one dimension at a time. Let $a := \epsilon - 2\delta$, $b := \epsilon + 2\delta$, $c := \epsilon + \delta$. We will proceed by induction on the dimension of the simplices.

We will start by showing that the 2-simplices are already (a, b, d_2) -good for some $d_2 := d_2(a, b)$.

Lemma 6. *A geodesic triangle in \mathbb{H}^2 with edge lengths in $[a, b]$ and circumradius at most R has altitudes bounded from below by a positive constant $h_0 := h_0(a, b, R)$.*

Proof. Since the sum of the angles of t are less than π there are at least two angles of t which are less than $\pi/2$. Let p be the vertex opposite these angles. The orthogonal projection of p onto the line containing the opposite edge $[q, r]$ is contained in the interior of $[q, r]$. Now suppose we have fixed the circumradius $r_0 \in [a/2, R]$ of t and consider all triangles with edges of length at least a such that p projects to the interior of $[q, r]$. The triangle with the shortest altitude at p is an isosceles triangle which lies on a hyperbolic circle of radius r_0 (see Figure 3). Let c be the center of the hyperbolic circle containing

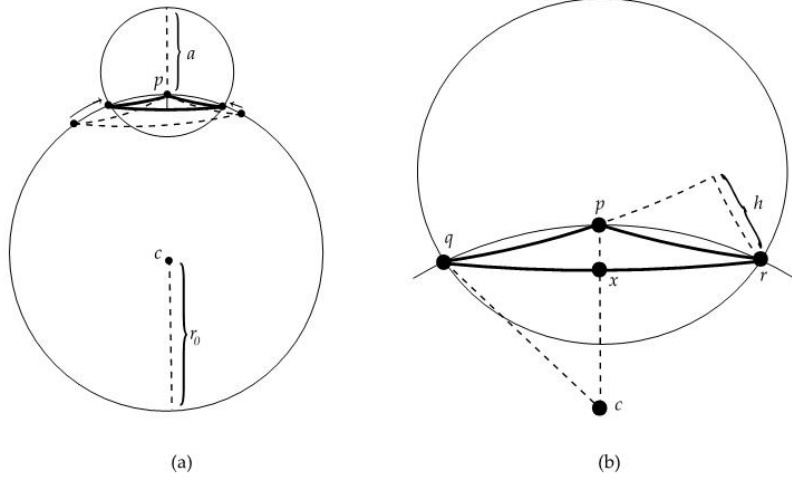


FIGURE 3. (a) If the radius r_0 is fixed, then moving q and r closer to p makes the altitude from p smaller. (b) Both $||[p, x]||$ and h are bounded from below in terms of a, b , and R .

p, q, r . Let x be the intersection of $[p, c]$ and $[q, r]$. Let $\alpha = \angle qpx$. Let $\beta = \angle pqx$.

By the hyperbolic law of cosines, we have

$$\begin{aligned} \cos(\alpha) &= \frac{\cosh(||[p, q]||) \cosh(||[p, c]||) - \cosh(||[q, c]||)}{\sinh(||[p, q]||) \sinh(||[p, c]||)} \\ &= \frac{\cosh(a) \cosh(r_0) - \cosh(r_0)}{\sinh(a) \sinh(r_0)}. \end{aligned}$$

By the hyperbolic law of sines we have $\sinh(||[q, x]||) = \sinh(||[p, q]||) \sin(\alpha)$. Now the altitude of $[p, q, r]$ from p is $||[p, x]||$. Using the law of cosines again, we get

$$\begin{aligned} \cosh(||[p, x]||) &= \cosh(a) \cosh(||[q, x]||) - \sinh(a) \sinh(||[q, x]||) \cos(\beta) \\ &\geq \cosh(a) \cosh(||[q, x]||) - \sinh(a) \sinh(||[q, x]||), \end{aligned}$$

Let $h_1(a, r_0) = \operatorname{arccosh}(\cosh(a) \cosh(||[q, x]||) - \sinh(a) \sinh(||[q, x]||))$. So far we have shown that the altitude from a vertex which projects to the interior of the opposite face is at least $h_1(a, r_0)$ if the circumradius of $[p, q, r]$ is r_0 . Since $h_1(a, r_0)$ decreases as r_0 increases, we have that $h_1(a, R)$ is a lower bound on the altitude from a vertex which projects to

the interior of the opposite face for triangles satisfying the hypotheses of the lemma. Let h' be the altitude from r .

We have

$$\begin{aligned}\sin(\beta) &= \frac{\sinh(||[p, x]||)}{\sinh(||[p, q]||)} \\ &\geq \frac{\sinh(h_1(a, R))}{\sinh(b)}.\end{aligned}$$

Also,

$$\begin{aligned}\sinh(h) &= \sinh(||[q, r]||) \sin(\beta) \\ &\geq \sinh(a) \cdot \frac{\sinh(h_1(a, R))}{\sinh(b)},\end{aligned}$$

so that

$$h \geq \operatorname{arcsinh}\left[\frac{\sinh(a)}{\sinh(b)} \cdot \sinh(h_1(a, R))\right].$$

A similar argument works for the altitude from q . Let

$$h_0(a, b, R) = \operatorname{arcsinh}\left[\frac{\sinh(a)}{\sinh(b)} \cdot \sinh(h_1(a, R))\right].$$

□

Since any 2-simplex in T has edge lengths in $[a, b]$ and circumradius at most c (where $a := \epsilon - 2\delta$, $b := \epsilon + 2\delta$, $c := \epsilon + \delta$), Lemma 6 implies that each 2-simplex in T is $(a, b, h_0(a, b, c))$ -good.

Assume that T_k is a $\frac{\delta}{100 \cdot 2^{k+1}}$ -good perturbation of T such that every simplex of dimension at most k which is contained in the μ -thick part of M is (a, b, d_k) -good for some positive constant $d_k \leq d_2$. Let $\delta_{k+1} := \frac{\delta}{100 \cdot 2^{k+1}}$. Let $d_{k+1} \leq d_k$ be a positive constant to be determined later.

Let $p_1 \in \mathcal{S} \cap M_{[\mu, \infty)}$. Let \mathcal{U}_1 be the set simplices $[v_0, \dots, v_l] \in \mathcal{T}_k$ of dimension at most k (i.e. $l \leq k$) such that there exists a δ_{k+1} -good perturbation \mathcal{T}' of \mathcal{T} which is obtained by perturbing only the point p_1 and such that $[p_1', v_0, \dots, v_l] \in \mathcal{T}'$.

By Lemma 4 and Lemma 5, the total volume of the (a, b, c, d_{k+1}) -bad regions of the l -simplices in \mathcal{U}_1 is bounded by $V_l(a, b, c, d_2, d_{k+1}) \cdot N(n, l, \mu)$, so that the total volume of the (a, b, c, d_{k+1}) -bad regions of all simplices in \mathcal{U}_1 is bounded by $\sum_{l=1}^{k+1} V_l(a, b, c, d_2, d_{k+1}) \cdot N(n, l, \mu)$. If we choose d_{k+1} so small that $\sum_{l=1}^{k+1} V_l(a, b, c, d_2, d_{k+1}) \cdot N(n, l, \mu) \leq \operatorname{vol}(B(p_1, \delta_{k+1}))$ (where $B(p_1, \delta_{k+1})$ is the ball of radius δ_{k+1} centered at p_1), then the (a, b, c, d_{k+1}) -bad regions of the simplices in \mathcal{U}_1 cannot cover $B(p_1, \delta_{k+1})$. Now choose

p_1' in $B(p_1, \delta_{k+1})$ (so that the perturbation is δ_{k+1} -good) and outside the (a, b, c, d_{k+1}) -bad region of every simplex in \mathcal{U}_1 . Call the new set of points \mathcal{S}_1 and the new triangulation \mathcal{T}_1 .

Assume we have perturbed the points p_1, \dots, p_s to p_1', \dots, p_s' and now have a set of points \mathcal{S}_s and a triangulation \mathcal{T}_s such that none of p_1', \dots, p_s' is the vertex of a (a, b, c, d_{k+1}) -bad simplex of dimension less than $k+2$. Let p_{s+1} be a point in $[\mathcal{S}_s \cap M_{[\mu, \infty)}] - \{p_1', \dots, p_s'\}$. Let \mathcal{U}_{s+1} be the set of simplices $[v_0, \dots, v_l] \in \mathcal{T}_s$ of dimension at most k such that there exists a δ_{k+1} -good perturbation \mathcal{T}_s' of \mathcal{T}_s which is obtained by perturbing only the point p_{s+1} and such that $[p_{s+1}', v_0, \dots, v_k] \in \mathcal{T}_s'$. Since d_{k+1} is so small, we can choose a point p_{s+1}' in the ball of radius δ_{k+1} centered at p_{s+1} and outside the (a, b, c, d_{k+1}) -bad region of every simplex in \mathcal{U}_{s+1} .

Assume that M has finite volume. Let \mathcal{T}' be the triangulation we get after perturbing every point of $\mathcal{S} \cap M_{[\mu, \infty)}$ once and only once (There are only finitely many since M has finite volume). Let $[p', v_1', \dots, v_l'] \in \mathcal{T}'$ be a simplex of dimension at most $k+1$. Suppose that p' was the last point perturbed among these $l+1$ vertices. We chose p' to be outside the (a, b, c, d_{k+1}) -bad region of $[v_1', \dots, v_l']$, so that $[p', v_1', \dots, v_l']$ is (a, b, d_{k+1}) -good. Thus any $(k+1)$ -simplex of \mathcal{T}' contained in $M_{[\mu, \infty)}$ is (a, b, d_{k+1}) -good.

If M has infinite volume, then for each positive integer m the above procedure can be used to perturb the vertices contained in an m -ball centered at some fixed point x_0 , giving us a geodesic triangulation \mathcal{T}_m of M such that any tetrahedron contained in $M_{[\mu, \infty)} \cap B(x_0, m)$ is (a, b, d_{k+1}) -good. Suppose we want to define the final triangulation on a ball $B(x_0, N)$ for some positive integer N . Since the triangulations \mathcal{T}_m agree on the ball $B(x_0, N)$ for $m \geq 100N$, we can use the triangulation \mathcal{T}_{100N} to define the triangulation inside $B(x_0, N)$.

We have shown that M has a geodesic triangulation such that every simplex of dimension at most $k+1$ which is contained in the thick part of M is (a, b, c, d_{k+1}) -good, completing the induction. When $k = n-1$, we get a geodesic triangulation of M such that every simplex of dimension at most n which is contained in the thick part of M is (a, b, d_n) -good. Thus the triangulation is L -thick for a constant L depending only on a, b , and d_n , which depend only on μ and n . \square

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